Exact Solution for some Nonlinear Partial Differential Equation which Describes Pseudo-Spherical Surfaces

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Signature
DEDICATION

To my dear parents, my wife, all my family, and to my friends.
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I would like to thank ...........

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ABSTRACT

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Nonlinear phenomena are very important in a variety of scientific fields. Finding solutions of nonlinear partial differential equations is one of the most difficult problems in mathematics and physics.

In this thesis, we study traveling wave solutions for Ibragimov-Shabat equation, Hunter – Saxton equation and Cassama – Holm equation, by using improved sine-cosine method, Wu’s elimination method, Tan-Cotan method and Sine-Cosine method, with the aid of Mathematica.
Introduction

Many partial differential equations which continue to be investigated due to their role in mathematics and physics exhibit interrelationships with the geometry of surfaces, or submanifolds, immersed in a three-dimensional space [1]. In particular, it has been known for a while that there is a relationship between surfaces of a constant negative Gaussian curvature in Euclidean three-space, the Sine-Gordon Equation and Bäcklund transformations which are relevant to the given equation [2]. Moreover, the original Bäcklund transformation for the Sine-Gordon Equation is also a simple geometric construction for pseudo-spherical surfaces [3]-[5]. It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from Physics to Biology, Chemistry, Mechanics, etc.

As mathematical models of the phenomena, the investigation of exact solutions to the NLPDEs reveals to be very important for the understanding of these physical problems. Many mathematicians and physicists have well understood this importance when they decided to pay special attention to the development of sophisticated methods for constructing exact solutions to the NLPDEs.

There are many methods used to solve nonlinear partial differential equations, such as, Bilinear method [6], which is developed by Hirota. It is the most efficient tool for seeking one solitons, or multisolitons for integrable nonlinear partial differential equations. The important step in the bilinear method is to transform the given equation
into Hirota bilinear form, and then may be find the bilinear form of a nonlinear partial differential equation or may be unable to find it. Who construct the soliton and multisoliton solutions of the nonlinear partial differential equation. Another method for solving the nonlinear partial differential equations is spectral method [7], who are numerical methods that give the approximate solution for the given equation with solutions that resemble shock waves. The approximate solution is very accurate and rapidly convergent with smooth solution [8, 9]. They provide an exponential convergence of the solution versus the number of collocation points.

Another method for solving nonlinear partial differential equations is Bäcklund transformation [10]; it was extended by M. Wang, Y. Wang and Y. Zhou [11, 12], which give the exact solution. It is based on generalized hyperbolic functions, (sinh, cosh, and tanh) and their properties. More methods dealing with the numerical solutions for nonlinear partial differential equations such as Sinc method which is developed by Frank Stenger [13]. It is used in various fields of numerical analysis, and solution of the nonlinear partial differential equations [14, 15].

The aim of this thesis is to establish exact solutions of distinct physical structures, solitons and kink waves solutions for the nonlinear partial differential equations.
Chapter One

Background Material

In this chapter, we discuss some topics in the applied mathematics that will be used in this thesis. The chain rule, ordinary differential equations, partial differential equations and the traveling wave solutions of partial differential equations, especially soliton and kink wave solutions will also be presented.

1.1 Differential Equations

We can classify the differential equations to:

a) Ordinary Differential Equation

Definition 1.1.1. [14]

A differential equation involving only ordinary derivative with respect to a single independent variable is called an Ordinary Differential Equation (ODE).

For example, Hermite equation

\[ y'' - 2xy' + 2y = 0 \ ; \ x \in \mathbb{R} \]

is an ordinary differential equation.

b) Partial Differential Equations

The subject of partial differential equations is one of the basic areas of applied analysis, and it is difficult to imagine any area of applications where its impact is not felt.
In the last few decades, there has been tremendous emphasis on understanding and modelling nonlinear processes; such processes are often governed by partial differential equations.

The subject of nonlinear partial differential equations has become one of the most active areas in applied mathematics and analysis.

**Definition 1.1.2 [14]**

A Partial Differential Equation (PDE) is an equation involving an unknown function of several variables and its partial derivatives.

For example, a second order PDE in two independent variables is an equation of the form:

$$U(x,t,y,y_x,y_t,y_{xx},y_{xt},y_{tt}) = 0; \quad (x,t) \in D$$  \hspace{1cm} (1.1.1)

Where the independent variables $x$ and $t$ lie in some given domain $D$ in $\mathbb{R}^2$ for some function $U$. When we solve (1.1.1), we mean a twice continuously differentiable function $u=u(x,t)$ defined on $D$, which when substituted into (1.2.1), reduces the equation to an identity on the domain $D$.

There are many types of solutions for PDE:

One of them when the function $u(x, t)$ is twice continuously differentiable, then $u(x, t)$ is called **Classical solution** or a **Genuine solution**.

Other type when the function $u(x, t)$ is discontinuous in its derivatives, then $u(x, t)$ is called **Weak solution**. For example, for the nonlinear equation $u_t + F(u)_x = 0$ (where $F(u)$ is a nonlinear convex function of $u(x, t)$, and $\{x \in \mathbb{R}, t \geq 0\}$), with bounded measurable initial data $u(x, 0)$, we say that the bounded measurable function $u=u(x, t)$ is a weak solution if

$$\int_{-\infty}^{\infty} \int_0^\infty [\phi u + \phi_x F(u)] dx dt = -\int_{-\infty}^{\infty} \phi(x,0) u(x,0) dx$$
holds for all test functions $\phi(x,t) \in C^1_0(\mathbb{R} \times \mathbb{R})$.

The domain $D$ in $\mathbb{R}^2$, where the problem is defined, is referred to as a space time domain, and the PDEs that include time $t$ as one of the independent variables are called

**Evolution equations**. When the two independent variables are both spatial variables, say $x$ and $y$ rather than $x$ and $t$, then the PDE will be called an **Equilibrium** or **Steady-state equation**.[27]

In the study of PDEs, the strategy is to classify them into different types depending either on the types of physical phenomena from which they arise, or on some mathematical basis. It is also important to classify PDEs as **Linear** or **Nonlinear**.

**Definition 1.1.3.**

An operator $L$ is called a **Linear operator** if for any two functions $u_1$ and $u_2$ and any constant $c$, we have

- $L(u_1 + u_2) = Lu_1 + Lu_2$, and
- $L(cu_1) = cLu_1$.

**Definition 1.1.4 :**

An equation of the form $Lu = F$, where $F = F(x, t)$, and $L$ is an operator, is **Linear Equation** if the operator $L$ is linear, otherwise the equation is nonlinear.

For example, the Laplace equation $u_{xx} + u_{yy} = 0$ is linear but the Burger’s equation $u_t + uu_x = \varepsilon u_{xx}$ is a nonlinear equation.

**Definition 1.1.5 :**[14]

The general form of a linear, second order PDE in two independent variables $x$, $y$ and the dependent variable $u(x, y)$ is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(1.1.2)

with $A,B,C,D,E,F$ and $G$ are constant. This equation is called:
**Elliptic** for \( B^2 - 4AC < 0 \)

**Parabolic** for \( B^2 - 4AC = 0 \)

**Hyperbolic** for \( B^2 - 4AC > 0 \)

**Example:**

- The wave equation \( u_{tt} - u_{xx} = 0 \) is hyperbolic.
- The Laplace equation \( u_{xx} + u_{yy} = 0 \) is elliptic.
- The heat equation \( u_t - u_{xx} = 0 \) is parabolic.

Now let us mention some of PDE which we will consider in this thesis Cassama–Holm equation, Hunter–Saxton equation and Ibragimov – Shabat equation

1) **Cassama–Holm equation:**

In fluid dynamics, the Cassama-Holm equation is the integrable dimensionless and non-linear partial differential equation

\[
u_t + 2\nu u_x - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 \tag{1.1.3}
\]

The equation was introduced by Camassa and Holm as a bi-Hamiltonian model for waves in shallow water.[28]

2) **Hunter–Saxton equation:**

In mathematical physics, the Hunter–Saxton equation

\[
u_{xxx} + 2u_xu_{xx} + uu_{xxx} = 0 \tag{1.1.4}
\]

is an integrable PDE that arises in the theoretical study of nematic liquid crystals. If the molecules in the liquid crystal are initially all aligned, and some of them are then wiggled slightly, this disturbance in orientation will propagate through the crystal, and the Hunter–Saxton equation describes certain aspects of such orientation waves.[29]
3) Ibragimov – Shabat equation

The evolution equation known as the Ibragimov – Shabat equation[26] is

\[-u_t + 3u^4u_x + 9uu_x^2 + 3u^2u_{xx} + uu_{xxx} = 0\]  \hspace{1cm} (1.1.5)

1.2 Traveling Wave Solutions

The theory of traveling wave solutions of parabolic equations is one of the fast
developing areas of modern mathematics. Traveling wave solutions of special type
which are characterized as solutions invariant with respect to translation in space. The
existence of traveling waves is very important in the study of nonlinear equations, and
they often determine the behavior of the solutions.

The reasons why the study of traveling wave solutions has become an important
part of the mathematical analysis of nonlinear processes are that: the analysis of
traveling waves provides a mean of finding explicit solutions of the equation. In
general, traveling wave solutions are easier to analyze. Other reason is that conform the
solutions with their natural occurrence in many mathematically modeled phenomena,
they characterize the long term behavior in numerous situations.

When we seek for traveling wave solutions for a partial differential equation of two
variables, we seek for solution of the form: \(u(x,t) = g(x - \lambda t) = g(\xi)\), where
\(\xi = x - \lambda t\), and \(\lambda\) denotes the speed of the wave.

In this way we reduce the partial differential equation to an ordinary differential
equation of the unknown function \(g(\xi)\) which is more easy to solve.

There are many types of traveling wave solutions, some of them are soliton and
kink waves solutions, which are solutions of specific nonlinear partial differential
equation.
Soliton solution is a wave that concave up or concave down, and it is vanishing function at the end points of the interval where the function is defined, that is, \( \lim_{x \to \pm \infty} u(x,t) = 0 \). So the boundary value of the soliton solution are the same, and this condition makes the solution of the equation is more easy. Other property of soliton wave is propagation, that is repeated itself for every equally subinterval of the hole interval in which the function is defined on. Soliton wave is smooth which makes the number of the critical points of the solution are few.

On the other hand, kink wave solution is similarly one of the types of traveling wave solutions of the nonlinear partial differential equations.

Kink wave is different from soliton wave, since it is not vanishing, so the kink wave is increasing or decreasing (which is known to be generalized kink). The kink wave may be symmetric about the y-axis (which is known to be compactons kink), where this two kinds of kink wave are different from each other.

These properties of soliton and kink waves determine the behavior of the solution, and hence the properties of the phenomena which the nonlinear partial differential equation is formed [17].

1.3 Curvature:

In mathematics, curvature is any of a number of loosely related concepts in different areas of geometry. Intuitively, curvature is the amount by which a geometric object deviates from being flat, or straight in the case of a line. but this is defined in different ways depending on the context. There is a key distinction between extrinsic curvature, which is defined for objects embedded in another space (usually a Euclidean space) in a way that relates to the radius of curvature of circles that touch the object, and intrinsic curvature, which is defined at each point in a Riemannian manifold.
**Principal Curvatures:**

At each point \( p \) of a differentiable surface in 3-dimensional Euclidean space one may choose a unit normal vector. A normal plane at \( p \) is one that contains the normal vector, and will therefore also contain a unique direction tangent to the surface and cut the surface in a plane curve, called normal section. This curve will in general have different curvatures for different normal planes at \( p \). The Principal Curvatures at \( p \), denoted \( k_1 \) and \( k_2 \), are the maximum and minimum values of this curvature. Here the curvature of a curve is by definition the reciprocal of the radius of the osculating circle. The curvature is taken to be positive if the curve turns in the same direction as the surface's chosen normal, and otherwise negative. The directions of the normal plane where the curvature takes its maximum and minimum values are always perpendicular, if \( k_1 \) does not equal \( k_2 \), a result of Euler (1760), and are called principal directions. From a modern perspective, this theorem follows from the spectral theorem because these directions are as the principal axes of a symmetric tensor—the second fundamental form. A systematic analysis of the principal curvatures and principal directions was undertaken by Gaston Darboux, using Darboux frames.

**Gaussian curvature:**

The Gaussian curvature or Gauss curvature of a surface at a point is the product of the principal curvatures, \( k_1 \) and \( k_2 \).

The sign of the Gaussian curvature can be used to characterize the surface.

- If both principal curvatures are the same sign: \( k_1 k_2 > 0 \), then the Gaussian curvature is positive and the surface is said to have an elliptic point. At such points the surface will be dome like, locally lying on one side of its tangent plane. All sectional curvatures will have the same sign.
• If the principal curvatures have different signs: $k_1k_2 < 0$, then the Gaussian curvature is negative and the surface is said to have a hyperbolic point. At such points the surface will be saddle shaped. For two directions the sectional curvatures will be zero giving the asymptotic directions.

• If one of the principal curvatures is zero: $k_1k_2 = 0$, the Gaussian curvature is zero and the surface is said to have a parabolic point.

Most surfaces will contain regions of positive Gaussian curvature (elliptical points) and regions of negative Gaussian curvature separated by a curve of points with zero Gaussian curvature called a parabolic line.
**Chapter Two**

**Pseudo-Spherical Surfaces**

The structure (Equations of pseudo-spherical type) was introduced by S.S. Chern and K. Tenenblat in 1986 [18], motivated by the fact that [19] generic solutions of equations integrable by the Ablowitz, Kaup, Newell and Segur (AKNS) inverse scattering scheme determine -whenever their associated linear problems are real-pseudo-spherical surfaces, that is Riemannian surfaces of constant Gaussian curvature $K = -1$.

**Definition 2.1.1: [20]**

A scalar differential equation $U(x, t, u, u_x, \ldots, u_{x^{n_m}}) = 0$ in two independent variables $x, t$ is of pseudo-spherical type (or, it is said to describe pseudo-spherical surfaces) if there exists one-forms $\omega^\alpha \neq 0$,

$$\omega^\alpha = f_{\alpha 1}(x, t, u, ..., u_{x^{\rho}})dx + f_{\alpha 2}(x, t, u, ..., u_{x^{\tau}})dt, \quad \alpha = 1, 2, 3 \quad (2.1.2)$$

whose coefficients $f_{\alpha \beta}$ are differential functions, such that the one-forms $\omega^\alpha$ satisfy the structure equations

$$d \omega^1 = \omega^3 \wedge \omega^2, \quad d \omega^2 = \omega^1 \wedge \omega^3, \quad d \omega^3 = \omega^1 \wedge \omega^2$$

whenever $u = u(x, t)$ is a solution to $U = 0$. 
We recall that a differential function is a smooth function which depends on $x$, $t$, and a finite number of derivatives of $u$. We sometimes use the expression “PSS equation” instead of “equation of pseudo-spherical type”. Also, we exclude from our considerations the trivial case when all functions $f_{\alpha\beta}$ depend only on $x$, $t$.

**Example 1.**

Burgers’ equation $u_t = u_{xx} + uu_x + h_\alpha(x)$ is a PSS equation with

$$\omega^1 = \left( \frac{1}{2} u - \frac{\beta}{\eta} \right) dx + \left( \frac{1}{2} u_x + \frac{1}{4} u^2 + \frac{1}{2} h(x) \right) dt,$$

$$\omega^2 = -\omega^3 = \eta dx + \left( \frac{\eta}{2} u + \beta \right) dt$$

in which $\eta \neq 0$ is a parameter, and $\beta$ is a solution of the equation

$$\beta^2 - \eta \beta + \frac{\eta^2}{2} h(x) = 0.$$ 

Now we want to show that Cassama–Holm equation, Hunter–Saxton equation and Ibragimov–Shabat equation are describes pseudo-spherical surfaces:

**a) Cassama–Holm equation:**

**Theorem 2.1.3:** [20]

The Cassama–Holm equation

$$m = u_{xx} - u, \quad m_x = -m_x u - 2m u_x$$

and Hunter–Saxton equation

$$m = u_{xx}, \quad m_x = -m_x u - 2m u_x$$

describe pseudo-spherical surfaces.

**Proof:** We consider one-forms $\omega^\alpha, \alpha = 1, 2, 3$, given by

$$\omega^1 = (m - \beta + \epsilon \eta^{-2} (\beta - 1)) dx + (-u_x \beta \eta^{-1} - \beta \eta^{-2} - u m - 1 + u \beta + u_x \eta^{-1} + \eta^{-2}) dt$$

$$\omega^2 = \eta dx + (-\beta \eta^{-1} - \eta u + \eta^{-1} + u_x) dt$$

$$\omega^3 = (m + 1) dx + (\epsilon u \eta^{-2} (\beta - 1) - um + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta^2} - \frac{u_x \beta}{\eta}) dt$$
in which the parameters $\eta$ and $\beta$ are constrained by the relation

$$\eta^2 + \beta^2 - 1 = \varepsilon \left( \frac{\beta - 1}{\eta} \right)^2$$

It is not hard to check that the structure equations (2.1.2) are satisfied whenever $u(x, t)$ is a solution of (Cassama–Holm) (if $\varepsilon = 1$ and $m = u_{xx} - u$ ) and whenever $u(x, t)$ is a solution of (Hunter–Saxton) (if $\varepsilon = 0$ and $m = u_{xx}$).

So by the last theorem, we obtained the Cassama–Holm equation

$$u_t + 2\tau u_x - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0$$

describe pseudo-spherical surfaces, with associated one-forms:

$$\omega^1 = (u_{xx} - u - \beta + \eta^{-2}(\beta - 1))dx + (-u_x \beta \eta^{-1} - \beta \eta^{-2} - u_{xx} + u_x + u^2 - 1 + u \beta - u_x \eta^{-1} + \eta^{-2})dt$$

$$\omega^2 = \eta dx + (-\beta \eta^{-1} - \eta u + \eta^{-1} + \eta u_x)dt$$

$$\omega^3 = (u_{xx} - u + 1)dx + (u \eta^{-2}(\beta - 1) - uu_{xx} + u_x + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta} - \frac{u_x \beta}{\eta})dt$$

whenever $u(x, t)$ is a solution of (Cassama–Holm equation).

b) Hunter–Saxton

By the last theorem, we obtained Hunter–Saxton equation

$$u_{xx} + 2u_x u_{xx} + uu_{xxx} = 0$$

describe pseudo-spherical surfaces, with associated one-forms:

$$\omega^1 = (u_{xx} - \beta)dx + (-u_x \beta \eta^{-1} - \beta \eta^{-2} - u_{xx} - 1 + u \beta + u_x \eta^{-1} + \eta^{-2})dt$$

$$\omega^2 = \eta dx + (-\beta \eta^{-1} - \eta u + \eta^{-1} + \eta u_x)dt$$

$$\omega^3 = (u_{xx} + 1)dx + (uu_{xx} + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta} - \frac{u_x \beta}{\eta})dt$$

whenever $u(x, t)$ is a solution of (Hunter–Saxton equation).
c) **Ibragimov – Shabat equation:**

The Ibragimov – Shabat equation

\[-u_t + 3u^5u_x + 9uu_x + 3u^2u_{xx} + uu_{xxx} = 0\]

describes pseudo-spherical surfaces, with associated one-forms:

\[
\omega^1 = \left(\frac{u_x}{u} + uu_x\right) dx + \left(\frac{u_{xxx}}{u} + uu_x^2 + 5uu_xu_{xx} + 9uu_x^3\right) dt
\]

\[
\omega^2 = \eta dx + \eta \left(\frac{u_{xx}}{u} + uu_x + 4uu_x\right) dt
\]

\[
\omega^3 = -\eta dx - \eta \left(\frac{u_{xx}}{u} + uu_x + 4uu_x\right) dt
\]

whenever \(u(x, t)\) is a solution of (Ibragimov – Shabat)[26].

As a consequence, each solution of the DE provides a local metric on \(M\), whose Gaussian curvature is constant, equal to \((-1)\).
Chapter Three

Some Methods to Solve Non-linear Partial Differential Equation

3.1 Improved sine-cosine method and Wu’s elimination method:

The main idea of the algorithm is as follows. Given a PDE of the form

\[ P(u, u_x, u_t, u_{xx}, u_{xxx}, \ldots) = 0 \]  \hspace{1cm} (3.1.1)

where \( P \) is a polynomial.

By assuming travelling wave solutions of the form

\[ u(x, t) = U(\rho), \quad \rho = \lambda (x - kt + c) \]  \hspace{1cm} (3.1.2)

where \( k, \lambda \) are parameters to be determined, and \( c \) is an arbitrary constant,

from the two equations (3.1.1) and (3.1.2) we obtain an ordinary differential equation

\[ Q(U, U', U'', \ldots) = 0 \]  \hspace{1cm} (3.1.3)

where \( U' = \frac{dU}{d\rho} \).

According to the improved sine-cosine method, we suppose that (3.1.3) has the following formal travelling wave solution

\[ U(\rho) = \sum_{i=1}^{n} \sin^{-1}(b_i \sin \psi + a_i \cos \psi) + a_0 \]  \hspace{1cm} (3.1.4)
and

\[ \frac{d\psi}{d\rho} = \sin \psi \quad \text{or} \quad \frac{d\psi}{d\rho} = \cos \psi \quad (3.1.5) \]

where \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) are constants to be determined. Then we proceed as follows:

(i) Equating the highest order nonlinear term and highest order linear partial derivative in (3.1.3), yield the value of \(n\).

(ii) Substituting (3.1.4), (3.1.5) into (3.1.1), we obtain a polynomial equation involving \(\cos \psi \sin^j \psi, \sin^j \psi\) for \(j = 0, 1, 2, \ldots, n\), (with \(n\) being positive integer).

(iii) Setting the constant term and coefficients of \(\sin \psi, \cos \psi, \cos \psi \sin \psi, \sin^2 \psi, \ldots\) in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers \(k, \lambda, a_0, a_i, b_j\) for \(i = 1, 2, \ldots, n\).

(iv) Using the Mathematica and the Wu’s elimination methods, the algebraic equations in (iii) can be solved.

These yield the solitary wave solutions for the system (3.1.1).[22]

We remarks that the above method yields solutions that includes terms sech \(\rho\) or \(\tanh \rho\), as well as their combinations. There are different forms of those obtained by other methods, such as the Homogenous Balance Method [23].

**Example:** mKdV equation:

**Modified Korteweg De Vries equation (mKdV):**

\[ u_t + q u^2 u_x + s u_{xxx} = 0 \quad (3.1.6) \]

To solve equation (3.1.6) by improved sine-cosine method and Wu’s elimination method, using the transformation,

\[ u(x,t) = U(\rho), \quad \rho = \lambda(x - kt + c) \quad (3.1.7) \]

where \(\lambda, k\) are constants to be determined later and \(c\) is an arbitrary constants.
Substituting (3.1.7) into (3.1.6), we obtain an ordinary differential:

\[ kU ' + qU^2U ' + s \lambda^2 U '''' = 0 \]  

(3.1.8)

where \( U ' = \frac{dU}{d\rho} \).

According to the algorithm of the improved sine-cosine method, we suppose that (3.1.8) has the following formal solutions

\[ U(\rho) = a_0 + b_1 \sin(\psi) + a_i \cos(\psi) \]  

(3.1.9)

and take \( \frac{d\psi}{d\rho} = \sin \psi \)  

(3.1.10)

with the aid of Mathematica, from (3.1.9) and (3.1.10), we get

\[
kU ' + qU^2U ' + s \lambda^2 U '''' = 2q \sin(\Psi) a_0 a_1 b_1 - 4q \sin^3(\Psi) a_0 a_1 b_1 + \sin^4(\Psi)(6s\lambda^2 a_1 + q a_1^3 - 3qa_1 b_1^2) + \sin^2(\Psi)(-ka_1 - 4s\lambda^2 a_1 - qa_0 a_1 - qa_1^3 + 2qa_1 b_1^2) + \\
\cos(\Psi)(s \lambda^2 b_1 + qa_0^2 b_1 + qa_1^2 b_1) + \sin^2(\Psi)(-2qa_0 a_1^2 + 2qa_0 b_1^2) + \\
\sin^3(\Psi)(-6s\lambda^2 b_1 - 3qa_1^2 b_1 + qb_1^3) = 0
\]

Setting the coefficients of \( \cos^i \psi \sin^j \psi \) for \( i=0,1 \) and \( j = 1 \) to 4, we have the following set of over determined equations in the unknowns \( k, \lambda, a_0, a_i \) and \( b_i \):

\[
\begin{align*}
2qa_0 a_1 b_1 &= 0, \\
-4qa_0 a_1 b_1 &= 0, \\
(6s\lambda^2 a_1 + qa_1^3 - 3qa_1 b_1^2) &= 0, \\
(-ka_1 - 4s\lambda^2 a_1 - qa_0 a_1 - qa_1^3 + 2qa_1 b_1^2) &= 0, \\
(kb_1 + s\lambda^2 b_1 + qa_0^2 b_1 + qa_1^2 b_1) &= 0, \\
(-2qa_0 a_1^2 + 2qa_0 b_1^2) &= 0, \\
(-6s\lambda^2 b_1 - 3qa_1^2 b_1 + qb_1^3) &= 0.
\end{align*}
\]

(3.1.11)

We now solve the above set of equations by using the Wu elimination method, and obtain the following solution:
By integrating $\frac{dy}{d\rho} = \sin \psi$ and taking the integration constant equal zero, we obtain

$$\sin \psi = \text{sech} \rho, \quad \cos \psi = \pm \tanh \rho \quad (3.1.13)$$

Substituting (3.1.12) and (3.1.13) into (3.1.9), we obtain

$$U(\rho) = \pm \sqrt{\frac{k - 2s \lambda^2}{q}} \pm \frac{6s}{q} \lambda \tanh(\rho)$$

where $\rho = \lambda(x - kt + c)$

### 3.2 Tan-Cot Method:

This method is applied to find out an exact solution of a nonlinear partial differential equation. Consider the following PDE,

$$P(u,u_x,u_t,u_{xt},u_{xxx},...)=0 \quad (3.2.1)$$

where $P$ is a polynomial of the variable $u$ and its derivatives. If we consider $u(x,t) = U(\xi)$, $\xi = k(x - \lambda t)$, where $k$ and $\lambda$ are real constants, so that we can use the following changes:

$$\frac{\partial}{\partial t} = -k \lambda \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = k \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = k^2 \frac{d^2}{d\xi^2}, \quad \frac{\partial^3}{\partial x^3} = k^3 \frac{d^3}{d\xi^3}$$

and so on, then (3.2.1) becomes an ordinary differential equation

$$Q(U,U',U'',U''',...)=0 \quad (3.2.2)$$

with $Q$ being another polynomial form of its argument, which will be called the reduced ordinary differential equation of (3.2.2). Integrating (3.2.2) as long as all terms contain
derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well [13].

The finding of the traveling wave solutions to (3.1.1) is equivalent to obtaining the solution of the reduced ordinary differential equation (3.1.2). Applying tan-cotan method, the solutions of nonlinear equations can be expressed as:

\[
 f(\xi) = a \tan^b(\mu \xi), |\xi| \leq \frac{\pi}{2\mu} \quad \text{or} \quad f(\xi) = a \cot^b(\mu \xi), |\xi| \leq \frac{\pi}{2\mu}
\]

where \(a\), \(\mu\), and \(b\) are parameters to be determined, \(\mu\) is the wave number. We use \(f(\xi)\) and their derivative:

\[
 f(\xi) = a \tan^b(\mu \xi)
\]

\[
 f'(\xi) = ab \mu \left( \tan^{b-1}(\mu \xi) + \tan^{b+1}(\mu \xi) \right)
\]

\[
 f''(\xi) = ab \mu^2 \left( (b - 1) \tan^{b-2}(\mu \xi) + 2b \tan^b(\mu \xi) + (b + 1) \tan^{b+2}(\mu \xi) \right)
\]

(3.2.3)

Or use

\[
 f(\xi) = a \cot^b(\mu \xi)
\]

\[
 f'(\xi) = -ab \mu \left( \cot^{b-1}(\mu \xi) + \cot^{b+1}(\mu \xi) \right)
\]

\[
 f''(\xi) = ab \mu^2 \left( (b - 1) \cot^{b-2}(\mu \xi) + 2b \cot^b(\mu \xi) + (b + 1) \cot^{b+2}(\mu \xi) \right)
\]

(3.2.4)

We substitute (3.2.3) or (3.2.4) into the reduced equation (3.2.2), balance the terms of the tan functions when (3.2.3) are used, or balance the terms of the cot functions when (3.2.4) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. Next collect all terms with the same power in \(\tan^k(\mu \xi)\) or \(\cot^k(\mu \xi)\) and set their coefficients to zero we get a system of algebraic equations among the unknown's \(a\), \(\mu\) and \(b\), and solve the subsequent system. [21]

**Example:** One-dimensional Burgers’ equation

Consider the one-dimensional Burgers’ equation which has the form

\[
 u_t + p u u_x - r u_{xx} = 0
\]

(3.2.5)
Where \( p \) and \( r \) are arbitrary constants. In order to solve (3.2.5) by the Tan method, we use the wave transformation
\[
u(x,t) = U(\xi), \quad \xi = (x - \lambda t)
\]
(3.2.5) takes the form of an ordinary differential equation.
\[-\lambda U' + p U U' - r U'' = 0 \quad (3.2.6)
\]
where \( U' = \frac{dU}{d\xi} \).

Integrating (3.2.6) once with respect to \( \xi \) and setting the constant of integration to be zero, we obtain:
\[-\lambda U + \frac{1}{2} p U^2 - r U' = 0 \quad (3.2.7)
\]

Substituting the solution in (3.2.3)
\[-\lambda a \tan^b(\mu \xi) + \frac{1}{2} p(a \tan^b(\mu \xi))^2 - r a b \mu \left(\tan^{b-1}(\mu \xi) + \tan^{b+1}(\mu \xi)\right) = 0 \quad (3.2.8)
\]
Equating the exponents and the coefficients of each pair of the tan functions we find the following algebraic system:

\[2b = 1 \Rightarrow b = 1 \quad (3.1.9)
\]

Substituting (3.2.8) into (3.1.9) to get: \( a = \frac{2r \mu}{p} \). (3.1.10)

Then by substituting (3.1.10) into (3.2.3) the solution of equation can be written in the form
\[u(x,t) = \frac{2r \mu}{p} \tan(\mu(x - \lambda t))) \quad (3.1.11)
\]
For \( \mu=\lambda = 1 \), \( r=0.5 \) and \( p=0.1 \), (3.1.11) becomes: \( u(x,t) = 10 \tan(x - t) \)
3.3 Sine-Cosine Method:

This method is applied to find out an exact solution of a nonlinear partial differential equation. Consider the following PDE,

\[ P(u, u_x, u_t, u_{xx}, u_{xxx}, \ldots) = 0 \]  \hspace{1cm} (3.3.1)

where \( P \) is a polynomial of the variable \( u \) and its derivatives.

1. To find the traveling wave solution of equation (3.3.1) we introduce the wave variable \( \xi = (x - ct + k) \) so that \( u(x,t) = U(\xi) \).

2. Based on this, we use the following changes:

\[
\frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}, \quad \frac{\partial^3}{\partial x^3} = \frac{d^3}{d\xi^3}
\]

and others for several derivatives. Then (3.3.1) becomes an ordinary differential equation

\[ Q(U, U_{\xi}, U_{\xi\xi}, U_{\xi\xi\xi}, \ldots) = 0 \]  \hspace{1cm} (3.3.2)

3. We then integrate the ordinary differential equation (3.3.2) as many times as possible, and setting the constant of integration to be zero.

4. Following the conclusions made in [25], the solutions may be set in the form

\[ u(x,t) = \lambda \sin^{\beta}\left(\mu \xi\right), \quad |\xi| \leq \frac{\pi}{\mu} \]  \hspace{1cm} (3.3.3)

or in the form

\[ u(x,t) = \lambda \cos^{\beta}\left(\mu \xi\right), \quad |\xi| \leq \frac{\pi}{\mu} \]  \hspace{1cm} (3.3.4)

where \( \lambda, \mu \) and \( \beta \) are parameters that will be determined.

5. As a consequence, the derivatives of (3.3.3) become

\[ u(x,t) = \lambda \sin^{\beta}\left(\mu \xi\right), \quad |\xi| \leq \frac{\pi}{\mu} \]

\[ u^n = \lambda^n \sin^{n\beta}\left(\mu \xi\right) \]  \hspace{1cm} (3.3.5)

\[ (u^n)_\xi = n \mu \beta \lambda^n \sin^{n\beta-1}\left(\mu \xi\right) \cos\left(\mu \xi\right) \]

\[ (u^n)_{\xi\xi} = n^2 \mu^2 \lambda^{n+1} \beta (n \beta - 1) \sin^{n\beta-2}\left(\mu \xi\right) - n^2 \mu^2 \beta^2 \lambda^{n+1} \sin^{n\beta}\left(\mu \xi\right) \]
and the derivative of (3.3.4) become

\[ u(x,t) = \lambda \cos^\beta (\mu \xi) \quad |\xi| \leq \frac{\pi}{\mu} \]

\[ u^n = \lambda^n \cos^{n\beta} (\mu \xi) \quad (3.3.6) \]

\[ (u^n)'_\xi = -n \mu \beta \lambda^n \cos^{n\beta-1} (\mu \xi) \sin (\mu \xi) \]

\[ (u^n)''_\xi = n \mu^2 \lambda^n \beta (n \beta - 1) \cos^{n\beta-2} (\mu \xi) - n^2 \mu^2 \beta^2 \lambda^n \cos^{n\beta} (\mu \xi) \]

and so on for other derivatives.

6. We substitute (3.3.5) or (3.3.6) into the reduced equation obtained above in (3.3.2), balance the terms of the cosine functions when (3.3.6) is used, or balance the sine functions when (3.3.5) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations to obtain all possible values of the parameters \( \lambda, \mu \) and \( \beta \).

The main advantage of this method is that it can be applied directly to most types of differential equations. Another important advantage is that it is capable of greatly reducing the size of computational work [24].

**Example:** Generalized KdV equation

Consider the following Generalized Korteweg De Vries equation (gKdV) equation

\[ u_t + (n + 1)(n + 2)u^n u_x + u_{xxx} = 0 \quad (3.3.7) \]

We now employ the sine–cosine method. Using the wave variable \( \xi = x - ct \) carries (3.3.7) into ODE

\[ -cu' + (n + 1)(n + 2)u^n u' + u''' = 0 \quad (3.3.8) \]

Integrating (3.3.8) gives and by considering the constant of integration to be zero for simplicity, we get
\[-cu + (n + 2)u^{n+1} + u'' = 0\]

(3.3.9)

Substituting (3.3.6) into (3.3.9) gives:

\[-c \lambda \cos \beta (\mu \xi) + (n + 2) \lambda^{n+1} \cos^{(n+1)\beta} (\mu \xi)\]

\[-\lambda \mu^2 \beta^2 \cos \beta (\mu \xi) + \lambda \mu^2 \beta (\beta - 1) \cos^{\beta-2} (\mu \xi) = 0\]

Equating the exponents and the coefficients of each pair of the cosine functions, we find the following system of algebraic equations:

\[
\begin{align*}
\beta - 1 & \neq 0 \\
(n + 1)\beta &= \beta - 2 \\
-c \lambda &= \lambda \mu^2 \beta^2 \\
(n + 2)\lambda^{n+1} &= -\lambda \mu^2 \beta (\beta - 1)
\end{align*}
\]

Solving the system yields:

\[
\begin{align*}
\beta &= \frac{-2}{n} \\
\lambda &= \left(\frac{1}{2} c\right)^{\frac{1}{n}} \\
\mu &= \frac{n}{2} \sqrt{-c}, \ c < 0
\end{align*}
\]

The result can be easily obtained if we also use the sine method (3.3.5). We obtain the following periodic solutions for \(c < 0\),

\[
u_1(x, t) = \left[\left(\frac{1}{2} c\right) \sec^2 \left(\frac{n}{2} \sqrt{-c} (x - ct)\right)\right]^{\frac{1}{n}}, \quad |\mu \xi| < \frac{\pi}{2}
\]

\[
u_2(x, t) = \left[\left(\frac{1}{2} c\right) \csc^2 \left(\frac{n}{2} \sqrt{-c} (x - ct)\right)\right]^{\frac{1}{n}}, \quad 0 < |\mu \xi| < \pi
\]

However, for \(c > 0\), we obtain the soliton solution

\[
u_3(x, t) = \left[\left(\frac{1}{2} c\right) \sec h^2 \left(\frac{n}{2} \sqrt{c} (x - ct)\right)\right]^{\frac{1}{n}}
\]
Chapter Four

Exact Solutions for some Nonlinear Partial Differential Equations

4.1 Cassama-Holm equation:

The Cassama-Holm equation is given by

\[ u_t + 2ru_x - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 \]  \hspace{1cm} (4.1.1)

Let \( u(x,t) = u(\xi) \) and by using the wave variable \( \xi = k(x - \lambda t) \) where \( k \) and \( \lambda \) are real constants. Equation (4.1.1) reduces to be the following ordinary differential equation:

\[ (2r - \lambda)u + \lambda k^2u'' + 3k uu' - 2k^3u' - k^3uu'' = 0 \]  \hspace{1cm} (4.1.2)

Integrating equation (4.1.2) once with zero integrating constant, we have

\[ (2r - \lambda)u + \lambda k^2u'' + \frac{3}{2}u^2 - \frac{1}{2}k^2(u')^2 - k^2uu'' = 0 \]  \hspace{1cm} (4.1.3)

Applying Cot-Csc Method, the solutions of nonlinear equations can be expressed as:
\( f(\xi) = a \cot^b(\mu \xi), |\xi| \leq \frac{\pi}{2\mu} \) where \( a, \mu, \) and \( b \) are parameters to be determined, \( \mu \) is the wave number.

We use \( f(\xi) \) and their derivative:

\[
\begin{align*}
  f'(\xi) &= -ab \mu \left( \cot^{b-1}(\mu \xi) + \cot^{b+1}(\mu \xi) \right), \\
  f''(\xi) &= ab \mu^2 \left( (b-1) \cot^{b-2}(\mu \xi) + 2b \cot^b(\mu \xi) + (b+1) \cot^{b+2}(\mu \xi) \right)
\end{align*}
\]

So the equation (4.1.3) becomes:

\[
(2r - \lambda)u + \lambda k^2 u'' + \frac{3}{2} u^2 - \frac{1}{2} k^2 (u^2)'' - k^2 uu'' = a(2r - \lambda) \cot^b(\mu \xi) + \frac{3}{2} a^2 \cot^{2b}(\mu \xi)
\]

\[-\frac{1}{2} a^2 b^2 k^2 \mu^2 (\cot^{-1b}(\mu \xi) + \cot^{1b}(\mu \xi))^2
\]

\[+abk^2 \lambda \mu^2 \left( (-1+b) \cot^{-2b}(\mu \xi) + 2b \cot^b(\mu \xi) + (1+b) \cot^{2b}(\mu \xi) \right)
\]

\[-a^2 bk^2 \mu^2 \cot^b(\mu \xi) \left( (-1+b) \cot^{-2b}(\mu \xi) + 2b \cot^b(\mu \xi) + (1+b) \cot^{2b}(\mu \xi) \right) = 0
\]

Equating the exponents and the coefficients of each pair of the cotan functions we find the following algebraic system:

\[
b + 2 = 2b \Rightarrow b = 2
\]

Substituting the value of \( b \) into (4.1.4):

\[
a(2r - \lambda) + 8ak^2 \lambda \mu^2 = 0
\]

\[
\frac{3}{2} a^2 - 12a^2 k^2 \mu^2 + 6ak^2 \lambda \mu^2 = 0
\]

\[
-2a^2 k^2 \mu^2 - 2a^2 k^2 \mu^2 = 0
\]

then we get the following system of equations:
\[ a(2r - \lambda) + 8ak^2\lambda \mu^2 = 0 \]
\[ \frac{3}{2}a^2 - 12a^2k^2\mu^2 + 6ak^2\lambda \mu^2 = 0 \]  
(4.1.6)

By solving (4.3.7) we find:

\[ \lambda = \frac{2r}{1-8k^2\mu^2}, \quad a = \frac{24k^2r\mu^2}{(1-8k^2\mu^2)(24k^2\mu^2-3)} \]  
(4.1.7)

Then the exact soliton solution of equation (4.3.1) can be written in the form:

\[ u(x,t) = \frac{24k^2\mu^2r}{(1-8k^2\mu^2)(24k^2\mu^2-3)}\cot^2\left(\mu k \left( x - \frac{2r}{1-8k^2\mu^2} t \right) \right) \]  
(4.1.8)

For \( \mu=k=r=1 \), (4.3.8) becomes:  
\[ u(x,t) = -\frac{24}{147}\cot^2\left( x + \frac{2}{7} t \right) \]

Figure 1: Travelling wave solutions of (4.1.1) at \( \mu=k=r=1 \).
4.2 Hunter - Saxton equation:

The Hunter - Saxton equation is given by

\[ u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0 \]  \hspace{1cm} (4.2.1)

By using Sine-Cosine method we introduce the wave variable \( \xi = k(x-ct) + \lambda \)
where \( k \neq 0 \), so that \( u(x,t) = U(\xi) \), and I obtained nonlinear ordinary differential equation in the form

\[-k^3 c u'' + 2k^3 u' u'' + k^3 u u''' = 0 \] \hspace{1cm} (4.2.2)

where \( \frac{du}{d\xi} = u' \).

Integrate the ordinary differential equation (4.2.2), and setting the constant of integration to be zero, we have

\[-k^3 c u'' + k^3 (u')^2 + k^3 (u u'' - \frac{1}{2} (u')^2) = 0 \] \hspace{1cm} (4.2.3)

\[ \frac{1}{2} k^3 (u')^2 + k^3 (u - c) u'' = 0 \] \hspace{1cm} (4.2.4)

Multiply (4.2.4) by \( \frac{1}{\sqrt{u-c}} \), we obtain:

\[ \frac{1}{\sqrt{u-c}} \left[ \frac{1}{2} k^3 (u')^2 + k^3 (u - c) u'' = 0 \right] \]

Or

\[ \frac{k^3}{2\sqrt{u-c}} (u')^2 + k^3 \sqrt{u-c} u'' = 0 \] \hspace{1cm} (4.2.5)

we can write (4.2.5) in the form
\[ k^3 [\sqrt{u - c} \ u'] = 0 \]  \hspace{1cm} (4.2.6)

where \( k \neq 0 \):

\[ [\sqrt{u - c} \ u'] = 0 \]  \hspace{1cm} (4.2.7)

by integrating (4.2.7), we obtain

\[ \sqrt{u - c} \ u' = c_1 \]  \hspace{1cm} (4.2.8)

\[ \Rightarrow \sqrt{u - c} \ \frac{du}{d\xi} = c_1 \]

\[ \sqrt{u - c} \ du = c_1 \ d\xi \]  \hspace{1cm} (4.2.9)

by integrating (4.2.8), we obtain

\[ \frac{2}{3} (u - c)^\frac{3}{2} = c_1 \xi + c_2 \]

\[ (u - c)^\frac{3}{2} = c_1 \xi + c_2 \]

\[ (u - c)^\frac{3}{2} = \frac{3}{2} c_1 \xi + \frac{3}{2} c_2 \]

\[ (u - c)^\frac{3}{2} = d_1 \xi + d_2 \], where \( d_1 = \frac{3}{2} c_1, d_2 = \frac{3}{2} c_2 \).

\[ u = (d_1 \xi + d_2)^\frac{2}{3} + c \]

We obtain the solution of Hunter - Saxton equation as the form

\[ u(x, t) = \left( d_1 (k (x - ct) + \lambda) + d_2 \right)^\frac{2}{3} + c \]

where \( d_1 \) and \( d_2 \) are constants.

If \( d_1 = d_2 = c = k = 1 \) and \( \lambda = 3 \) then \( u(x, t) = (x - t + 4)^\frac{2}{3} + 1 \)
Figure 2: Solution $u$ is shown at $t = 0, 1, 2, d_1 = d_2 = c = k = 1$ and $\lambda = 3$.

Figure 3: Travelling wave solutions of (4.2.1) at $d_1 = d_2 = c = k = 1$ and $\lambda = 3$. 
4.3 Ibragimov – Shabat Equation:

The Ibragimov – Shabat equation is given by

\[-u_t + 3u^4u_x + 9uu_x^2 + 3u^2u_{xx} + u_{xxx} = 0\] (4.3.1)

To solve equation (4.3.1) by Improved sine-cosine method and Wu’s elimination method, using the transformation, \(u(x, t) = U(\rho), \ \rho = \lambda(x - kt + c)\)

So we use the following changes:

\[\frac{\partial u}{\partial t} = -k\lambda \frac{dU}{d\xi}, \ \frac{\partial u}{\partial x} = \lambda \frac{dU}{d\xi}, \ \frac{\partial^3 u}{\partial x^3} = \lambda^2 \frac{d^3U}{d\xi^2}, \ \frac{\partial^3 u}{\partial x^3} = \lambda^3 \frac{d^3U}{d\xi^3}\]

Substituting these derivatives into equation (4.3.1) results into an ordinary differential equation of the form

\[c \frac{dU}{d\xi} + 3U^4(\xi)\frac{dU}{d\xi} + 9U(\xi)\left(\frac{dU}{d\xi}\right)^2 + 3U^2(\xi)\frac{d^2U}{d\xi^2} + \frac{d^3U}{d\xi^3} = 0\] (4.3.2)

\[-k\lambda U' - 3\lambda U^3U' - 9\lambda^2U(U')^2 - 3\lambda^2U^2U'' - \lambda^3U''' = 0\]
\[kU' + 3U^2U' + 9\lambda U(U')^2 + 3\lambda U^2U'' + \lambda^2U''' = 0\] (4.3.3)

where \(U' = \frac{dU}{d\rho}\).

I) we suppose that equation (4.1.3) has the following formal solutions:

\[U(\rho) = a_0 + b_1 \sin(\psi) + a_1 \cos(\psi)\] (4.3.4)

and
\[
\frac{d\psi}{d\rho} = \sin\psi
\]  

(4.3.5)

so

\[
U(\psi) = a_0 + b_1 \sin(\psi) + a_1 \cos(\psi)
\]

(4.3.3)

\[
U'(\psi) = b_1 \cos(\psi) \sin(\psi) - a_1 \sin^2(\psi)
\]

(4.3.6)

\[
\begin{align*}
U''(\psi) &= b_1 \cos^2(\psi) \sin(\psi) - b_1 \sin^3(\psi) - 2a_1 \cos(\psi) \sin^2(\psi) \\
U'''(\psi) &= -4a_1 \cos^2(\psi) \sin^2(\psi) + 2a_1 \sin^4(\psi) + b_1 \sin(\psi) \cos^3(\psi) - 5b_1 \cos(\psi) \sin^3(\psi)
\end{align*}
\]

II) From (4.3.3) and (4.3.6), we get

\[
kU' + 3U'U' + 9\lambda U(U')^2 + 3\lambda U^2U'' + \lambda^2 U''' = \sin^6(\psi) \left(-3a_1^5 + 30a_1^3 b_1^2 - 15a_1 b_1^4\right)
\]

\[
+ \sin(\psi) \left(3\lambda a_0^2 b_1 + 12a_0^3 a_1 b_1 + 3\lambda a_1^2 b_1 + 12a_0 a_1^2 b_1\right)
\]

\[
+ \sin(\psi) \cos(\psi) \left(kb_1 + \lambda^2 b_1 + 3a_0^3 b_1 + 6\lambda a_0 a_1 b_1 + 18a_0^2 a_1^2 b_1 + 3a_1^2 b_1\right)
\]

\[
+ \sin^2(\psi) \left(-ka_1 - 4\lambda^2 a_1 - 3a_0^3 a_1 - 12\lambda a_0 a_1^2 - 18a_0^2 a_1^3 - 3a_1^3 + 9a_0 b_1^2 + 6\lambda a_0 b_1^2 + 36a_0^2 b_1^2 + 12a_1^2 b_1^2\right)
\]

\[
+ \cos(\psi) \sin^2(\psi) \left(-6\lambda a_0^2 a_1 - 12a_0^3 a_1^2 - 6\lambda a_1^3 - 12a_0 a_1^4 + 12a_0^2 b_1^2 + 9a_1 b_1^2 + 6\lambda a_1 b_1^2 + 36a_0 a_1^2 b_1^2\right)
\]

\[
+ \sin^3(\psi) \left(-6\lambda a_0^2 b_1 - 24a_0^3 a_1 b_1 - 18a_0^2 a_1^2 b_1 - 21\lambda a_0 a_1^3 b_1 - 60a_0^2 a_1^4 b_1 + 9b_1^3 + 3\lambda b_1^3 + 36a_0 a_1 b_1^3\right)
\]

\[
+ \cos(\psi) \sin^3(\psi) \left(-6\lambda^2 b_1 - 18a_0 a_1 b_1 - 24\lambda a_0 a_1^2 b_1 - 54a_0^2 a_1^2 b_1 - 18a_1 b_1 + 18a_0^2 b_1^3 + 18a_1^2 b_1^3\right)
\]

\[
+ \sin^4(\psi) \left(6\lambda^2 a_1 + 9a_0 a_1^2 + 12\lambda a_0 a_1^2 + 18a_0^2 a_1^3 + 6a_1^5 - 9a_0 b_1^2 - 12\lambda a_0 b_1^2 - 54a_0^2 a_1 b_1^2 - 42a_1^2 b_1^2 + 12a_1 b_1^4\right)
\]

\[
+ \sin^4(\psi) \cos(\psi) \left(9a_1^3 + 6\lambda a_1^2 + 12a_0 a_1^4 - 27a_1 b_1^2 - 18\lambda a_1 b_1^2 - 72a_0 b_1^2 b_1^2 + 12a_0 b_1^4\right)
\]
Setting the coefficients of \( \cos^i \psi \) \( \sin^j \psi \) for \( i = 0, 1 \) and \( j = 1 \) to \( 6 \), we have the following set of over determined equations in the unknowns \( k, \lambda, a_0, a_i \) and \( b_i \):

\[
\begin{align*}
(-3a_1^4 + 30a_1^2b_1^2 - 15b_1^4) &= 0 \\
(3\lambda a_0^2 + 12a_0^3a_1 + 3\lambda a_1^2 + 12a_0a_1^3) &= 0 \\
(k + \lambda^2 + 3a_0^4 + 6\lambda a_0a_1 + 18a_0^2a_1^2 + 3a_1^4) &= 0 \\
-k a_1 - 4\lambda^2 a_1 - 3a_0^4 a_1 - 12\lambda a_0a_1^3 - 18a_0^2a_1^3 - 3a_1^5 + 9a_0b_1^2 + 6\lambda a_0b_1^2 + 36a_0^2a_1b_1^2 + 12a_1^3b_1^2) &= 0 \\
(-6\lambda a_0^2a_1 - 12a_0^3a_1^2 - 6\lambda a_1^3 - 12a_0a_1^4 + 12a_0^3b_1^2 + 9a_1b_1^2 + 6\lambda a_1b_1^2 + 36a_0a_1^2b_1^2) &= 0 \\
(-6\lambda a_0^2b_1 - 24a_0^3a_1b_1 - 18a_1^2b_1^2 - 21\lambda a_1^3b_1 - 60a_0a_1^3b_1 + 9b_1^3 + 3\lambda b_1^3 + 36a_0a_1b_1^3) &= 0 \\
(-6\lambda^2 - 18a_0a_1 - 24\lambda a_0a_1 - 54a_0^2a_1^2 - 18a_1^4 + 18a_0^2b_1^2 + 18a_1^2b_1^2) &= 0 \\
(6\lambda^2a_1 + 9a_0a_1^2 + 12\lambda a_0a_1^2 + 18a_0^2a_1^3 + 6a_1^5 - 9a_0b_1^2 - 12\lambda a_0b_1^2 - 54a_0^2a_1b_1^2 - 42a_1^2b_1^2 + 12a_1b_1^4) &= 0 \\
(9a_1^3 + 6\lambda a_1^3 + 12a_0a_1^4 - 27a_1^2b_1^2 - 18\lambda a_1b_1^2 - 72a_0a_1^2b_1^2 + 12a_0b_1^4) &= 0 \\
(27a_1^2 + 18\lambda a_1^2 + 48a_0a_1^3 - 9b_1^2 - 6\lambda b_1^2 - 48a_0a_1b_1^2) &= 0 \\
(15a_1^4 - 30a_1^2b_1^2 + 3b_1^4) &= 0 
\end{align*}
\]
IV) We now solve the above set of equations by using Mathematica and the Wu’s elimination method, and obtain the following solution:

\[ a_1 = \frac{-\lambda}{4a_0} \]

\[ b_1 = \frac{-\lambda}{12.3107 a_0} \]

\[ a_0 = \frac{-(5\lambda^2 + 8k)^{\frac{1}{2}}(\lambda^2 + 4k)^{\frac{1}{2}}}{2^{\frac{1}{2}}} \] (4.3.8)

By integrating \( \frac{d\psi}{d\rho} = \sin\psi \) and taking the integration constant equal zero, we obtain

\[ \sin\psi = \text{sech} \rho , \quad \cos\psi = \pm \tanh \rho \] (4.3.9)

Substituting (4.3.8) and (4.3.9) into (4.3.4), we obtain

\[ U(\rho) = a_0 + b_1 \sin(\psi) + a_1 \cos(\psi) \]

\[ U(\rho) = \frac{-(5\lambda^2 + 8k)^{\frac{1}{2}}(\lambda^2 + 4k)^{\frac{1}{2}}}{2^{\frac{1}{2}}} + \frac{-\lambda}{12.3107 a_0} \text{sech}\rho \pm \frac{-\lambda}{4a_0} \tanh\rho \] (4.3.10)

where \( \rho = \lambda(x - kt + c) \)

For \( \lambda = 1, k = -1 \) and \( c = 1 \), (4.3.10) becomes

\[ u(x,t) = \frac{1}{2} - \frac{2}{12.3107} \text{sech}(x + t + 1) - \frac{1}{2} \tanh(x + t + 1) \]
Figure 4: Solution $u$ is shown at $t = 0, 1, 2$, $\lambda = 1, k = -1$ and $c = 1$.

Figure 5: Travelling wave solutions of (4.3.1) at $\lambda = 1, k = -1$ and $c = 1$. 
Conclusions

In this thesis, we considered the construction of exact solutions to some NLPDEs. and we study traveling wave solutions for some NLPDEs Ibragimov – Shabat equation, Hunter - Saxton equation and Cassama-Holm equation. We found that the improved sine cosine method is the suitable method which gives a non-trivial solution to Ibragimov – Shabat equation, while other methods lead to difficulty in integration. Solving Hunter - Saxton equation needs a specific transformation to transform it to an ordinary differential equation.

We recommend using other methods like improved exponential method in spite of needing new software like maple.
References


[16] Salas (1986), Calculus: One and Several Variables, with Analytic Geometry, **John Wiley and Sons, New York**.


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الحل الدقيق لبعض المعادلات التفاضلية الجزئية غير الخطية التي تصف السطوح كروية المحلة

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الملخص
المعادلات التفاضلية الجزئية غير الخطية تستخدم لوصف مجموعة كبيرة من الظواهر الطبيعية وعلوم الحياة كذلك علوم الطبيعة وعلوم الأرض والهندسة، وخلال هذه الظواهر شكلت ظاهرة الموجة غير الخطية أهمية كبيرة، وأخذت مساحة مؤثرة جداً في مجال البحث والدراسة خلال السنوات العشرين الماضية، ومن أهم هذه الظواهر الموجية مفهوم الحلول الموجية بشكل عام والتي منها الموجات السوليتونية والحلول السوليتونية بشكل خاص، حيث إنه جذب انتباه الكثير من الباحثين في جميع العلوم الحديثة مثل الرياضيات والفيزياء والهندسة وعلم الأحياء. وفي هذه الرسالة قمنا باستخدام بعض طرق حل المعادلات التفاضلية الجزئية غير الخطية لتقديم حلول دقيقة لبعض المعادلات مثل معادلة إبراجيموف ومعادلة هنتر ومعادلة كساما والتي تصف السطوح الكروية المحلة.